Information Structures and the Value of Information

Information Structures

The Representation of Information

- What is information? Everything comes from the description of a *state of nature* which is a full description of a contingency.
- For example, "it is raining today" or "it is sunny today."
- Or more precise statements of the type "the temperature is between 12 and 15 degrees celsius."
- A state of nature is denoted with $\omega \in \Omega$.
- We also assume a probability distribution (prior) π over all states of nature.
- There are two equivalent representations of information structures: through a *partition* or through *signals*.

The Partition Approach

- Every player has a partition P_i over the set of states.
- If two states ω_1, ω_2 belong to the same block in the partition, they are indistinguishable.
- For example, suppose that the true temperature is 14 degrees celsius.
- Agent 1 cannot distinguish between $12, 13, 14$ and 15 degrees celsius, $P_1 = \{\{12, 13, 14, 15\}\}$.
- Agent 2 cannot distinguish between $13, 14, 15, 16$ and 17 degrees celsius, $P_2 = \{\{13, 14, 15, 16, 17\}\}$.

Example

- The set $\{1, 2, 3\}$ has the following five partitions.
- $\{\{1\}, \{2\}, \{3\}\}$ - This is the *finest* partition.
- $\{\{1, 2\}, \{3\}\}$
- $\{\{1, 3\}, \{2\}\}$
- $\{\{1\}, \{2, 3\}\}$
- $\{\{1, 2, 3\}\}$ - This is the *coarsest* partition.

A Comparison of Partitions

- A partition P is *finer* than a partition P' if, whenever two states are in the same block in P, they are in the same block in P'.
- A partition P is *coarser* than a partition P' if, whenever two elements belong to different blocks in P, they belong to different blocks in P'.
- Thus, "finer" and "coarser" partitions are *partial orders* and not *complete orders*.
- How about the following partitions?
  $P_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$, $P_2 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$.

The Signal Approach

- Every player receives a signal which is correlated with the state.
- This approach is *equivalent* to the partition approach.
- In a finite world, what matters is the probability of signal i given state j, $p_{i|j}$.

Actions, States and Consequences

- There is a set of actions $X = \{1, ..., n\}$.
- There is a set of states of the world $S = \{1, ..., k\}$ with probability π_s.
- There is a set of consequences C_{xs}.
- An agent chooses the action x which maximizes $U(x) = \sum_s \pi_s v(c_{xs})$.

Messages, Prior and Posterior Beliefs

- There are five different probabilities:

1. The unconditional probability of state s, π_s,
2. The unconditional probability of receiving message m, q_m,
3. The joint probability distribution of state s and message m, j_{ms},
4. The (likelihood) conditional probability of message m given s, $q_{m|s}$,
5. The posterior probability of state s given message m, $\pi_{s|m}$.

Joint, Likelihood and Posterior Probabilities

- Let $J = [j_{ms}]$ be the matrix of joint probabilities.
- $\sum_{s,m} j_{ms} = 1$.
- $\sum_m j_{ms} = \pi_s$.
- $\sum_s j_{ms} = q_m$.
- Let $L = [q_{m|s}]$ be the likelihood matrix with $q_{m|s} = \frac{j_{ms}}{\pi_s}$.
- $\sum_m q_{m|s} = 1$.
- Let $\Pi = [\pi_{s|m}]$ be the posterior matrix with $\pi_{s|m} = \frac{j_{ms}}{q_m}$.
- $\sum_s \pi_{s|m} = 1$.

Example: Oil Drilling

- There might be oil under your land.
- Actions: you can either invest or not invest.
- There are three underlying states: state 1, state 2 and state 3.
- You drill to acquire information; there are two possible outcomes "wet" and "dry."
- If the true state is 1, wet appears with probability 90%, if it is 2 wet appears with probability 30%, and if it is 3, wet never appears.
- The initial probabilities for the three states are $(0.1, 0.5, 0.4)$.

Example: Oil Drilling - The Matrix L

State	Wet	Dry	
1	0.9	0.1	1.0
2	0.3	0.7	1.0
3	0	1.0	1.0

- Recall that $L = [q_{m|s}]$ is the likelihood matrix with $q_{m|s} = \frac{j_{ms}}{\pi_s}$.

Example: Oil Drilling - The Matrix J

State	Wet	Dry	
1	0.09	0.01	0.1
2	0.15	0.35	0.5
3	0	0.4	0.4
q_m	0.24	0.76	1.0

- Therefore $q_{m|s} \times \pi_s = j_{ms}$, where $\pi_1 = 0.1, \pi_2 = 0.5, \pi_3 = 0.4$.

Example: Oil Drilling - The Matrix Π

State	Wet	Dry
1	0.375	0.013
2	0.625	0.461
3	0	0.526

- Recall that $\Pi = [\pi_{s|m}]$ is the posterior matrix with $\pi_{s|m} = \frac{j_{ms}}{q_m}$, where q_m is the last row of Matrix J.

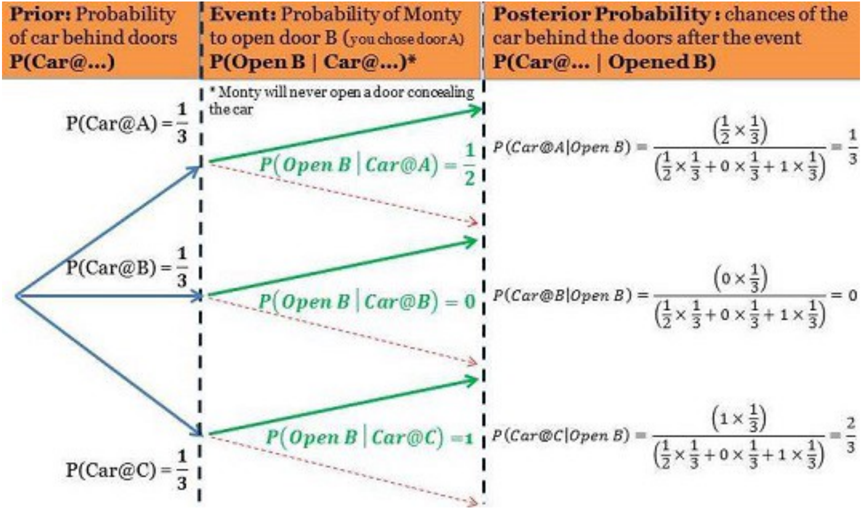
Example: The Monty Hall Game Show

- A contestant on a television game show may choose any of three curtained booths one of which contains a car.
- The contestant arbitrarily selects booth 1.
- Before the curtain is drawn, the presenter draws the curtain on another booth, booth 2, which is revealed to be empty.
- The contestant can choose to switch. Should the contestant switch?

Example: The Game Show - Resolution

State of the world	Prior	Likelihood	Joint	Posterior
Prize in 1	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$
Prize in 2	$\frac{1}{3}$	0	0	0
Prize in 3	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{2}{3}$

Pictorial Resolution



The Value of Information

The Value of Information

- Let x_0 be the optimal action which maximizes $U(x) = \sum_s \pi_s v(c_{xs})$.
- After receiving message m, the decision maker updates his beliefs to $\pi_{s|m}$.
- This will lead the decision maker (DM) to recalculate an optimal action x_m.
- The *value of information* is $\omega_m = U(x_m, \pi_{s|m}) - U(x_0, \pi_{s|m})$.
- The value of information is computed using the posterior probability; it is always nonnegative (the DM could always choose x_0).

Value of Information Structures

- In general, we want to compute the value of an information structure or message service.
- What matters is not ω_m but the expected utility over all possible messages m,

$$\Omega(\mu) = E_m\omega_m = \sum_m q_m[U(x_m, \pi_{.|m}) - U(x_0, \pi_{.|m})],$$

- where μ, the message service, is characterized by the likelihood matrix L and the vector of priors π.

Value of Information Structures (Cont.)

- If c^*_{sm} is the outcome of the best action at state s with message m, and c^*_{s0} is the outcome of the best action with the prior, the value of information is

$$\begin{aligned}\Omega(\mu) &= \sum_m q_m \sum_s \pi_{s|m} v(c^*_{sm}) - \sum_m \sum_s \pi_{s|m} q_m v(c^*_{s0}) \\ &= \sum_m \sum_s \pi_{s|m} q_m v(c^*_{sm}) - \sum_s \pi_s v(c^*_{s0}).\end{aligned}$$

Example: Oil Drilling Revisited

- Let's simplify the example. Assume there are only two states: wet (probability $\pi_1 = 0.24$) and dry (probability $\pi_2 = 0.76$).
- If I take action 1 (drill) and it is wet, then I get $\$1,000,000$; if it is dry, then I lose $\$400,000$.
- If I take action 2 (not drill) I incur a cost of $\$50,000$.
- Utility is linear $v(c) = c$.
- With the prior probabilities, action 1 gives a payoff of $1,000,000 * 0.24 - 400,000 * 0.76 = -64,000$.
- So the best action is action 2 with a value of $-50,000$.

Example: Oil Drilling with Messages

- Suppose you receive a message from a geological survey.
- The message service has the following likelihood (m: message; s:state).

	m Wet	m Dry
s Wet	0.6	0.4
s Dry	0.2	0.8

- Recall that $L = [q_{m|s}]$ is the likelihood matrix with $q_{m|s} = \frac{j_{ms}}{\pi_s}$, where $\pi_w = 0.24, \pi_d = 0.76$.

Example: Oil Drilling Optimal Choice with Messages

- The Matrix J is now

	m Wet	m Dry
s Wet	0.144	0.096
s Dry	0.152	0.608
q_m	0.296	0.704

- Therefore $j_{ms} = q_{m|s} \times \pi_s$, where $\pi_w = 0.24, \pi_d = 0.76$.

Example: Oil Drilling Optimal Choice with Messages (Cont.)

- The Posterior Matrix is now

	m Wet	m Dry
s Wet	0.486	0.136
s Dry	0.514	0.864

- Recall that $\Pi = [\pi_{s|m}]$ is the posterior matrix with $\pi_{s|m} = \frac{j_{ms}}{q_m}$, where q_m is the last row of Matrix J.

Example: Oil Drilling Optimal Choice with Messages (Cont.)

- The Posterior Matrix is

	m Wet	m Dry
s Wet	0.486	0.136
s Dry	0.514	0.864

- If message is wet, action 1 gives $0.486 * 1,000,000 - 0.514 * 400,000 = 280,400 > -50,000$, so action 1 is best.
- If message is dry, action 1 gives $0.136 * 1,000,000 - 0.864 * 400,000 < -50,000$ so action 2 is best.

Example: Oil Drilling Optimal Choice with Messages (Cont.)

- The Posterior Matrix is

	m Wet	m Dry
s Wet	0.486	0.136
s Dry	0.514	0.864

- The value of information is
$\Omega(\mu) = 0.296 * (280,400 - (-50,000)) + 0.704 * (-50,000 - (-50,000)) = 97,798.4$